RECURSION

CS10003: PROGRAMMING AND DATA STRUCTURES

Recursion

A process by which a function calls itself repeatedly.

- Either directly.
    - F calls F.
- Or cyclically in a chain.
    - F calls G, G calls H, and H calls F.

Used for repetitive computations in which each action is stated in terms of a previous result.

fact(n) = n * fact (n-1)

Basis and Recursion

For a problem to be written in recursive form, two conditions are to be satisfied:

- **It should be possible to express the problem in recursive form.**
- **The problem statement must include a stopping condition**

```
fact(n)  =  1,                    if  n = 0        /* Stopping criteria */
         =  n * fact(n – 1),      if  n > 0        /* Recursive form */
```

Examples:

- **Factorial:**

  fact(0) = 1
  fact(n) = n * fact(n – 1), if n > 0

- **GCD:**

  gcd (m, m) = m
  gcd (m, n) = gcd (m%n, n), if m > n
  gcd (m, n) = gcd (n, n%m), if m < n

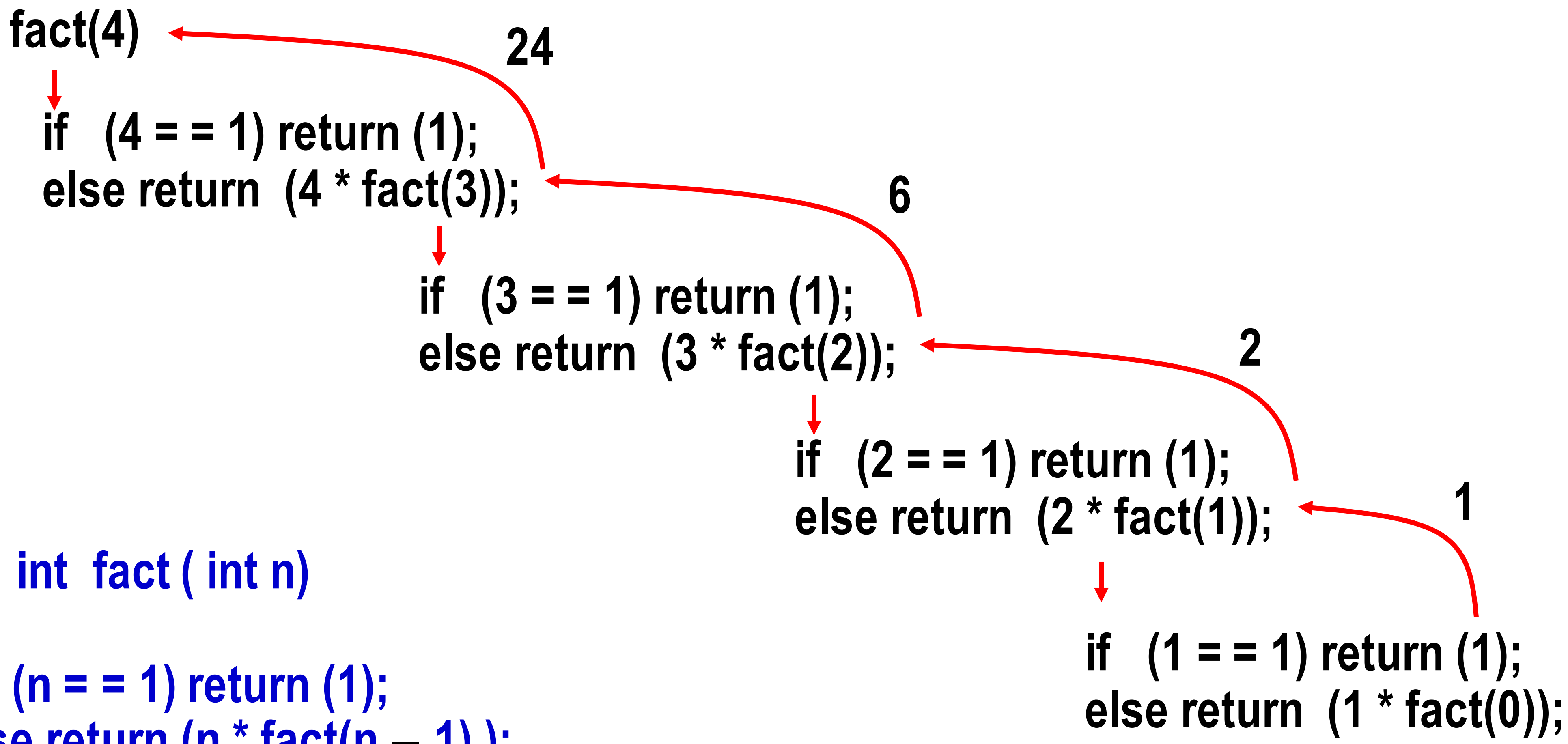
- **Fibonacci series (1,1,2,3,5,8,13,21,….)**

  fib (0) = 1
  fib (1) = 1
  fib (n) = fib (n – 1) + fib (n – 2), if n > 1

Example 1 :: Factorial

```
long  int  fact ( int n)
{
    if   (n = = 1)
        return (1);
    else
        return  (n * fact(n – 1));
}
```

Example 1 :: Factorial Execution



fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));
```

```
if   (3 = = 1) return (1);
else return  (3 * fact(2));
```

```
if   (2 = = 1) return (1);
else return  (2 * fact(1));
```

```
if   (1 = = 1) return (1);
else return  (1 * fact(0));
```

```
long  int  fact ( int n)
{
   if   (n = = 1) return (1);
   else return (n * fact(n – 1) );
}
```

Example 2 :: Fibonacci number

Fibonacci number f(n) can be defined as:

$f(0) = 0$

$f(1) = 1$

$f(n) = f(n-1) + f(n-2), \quad \text{if } n > 1$

- The successive Fibonacci numbers are:

  0, 1, 1, 2, 3, 5, 8, 13, 21, …..

Function definition:

```
int   f (int n)
{
    if  (n  < 2)   return (n);
    else  return ( f(n – 1) + f(n – 2) );
}
```

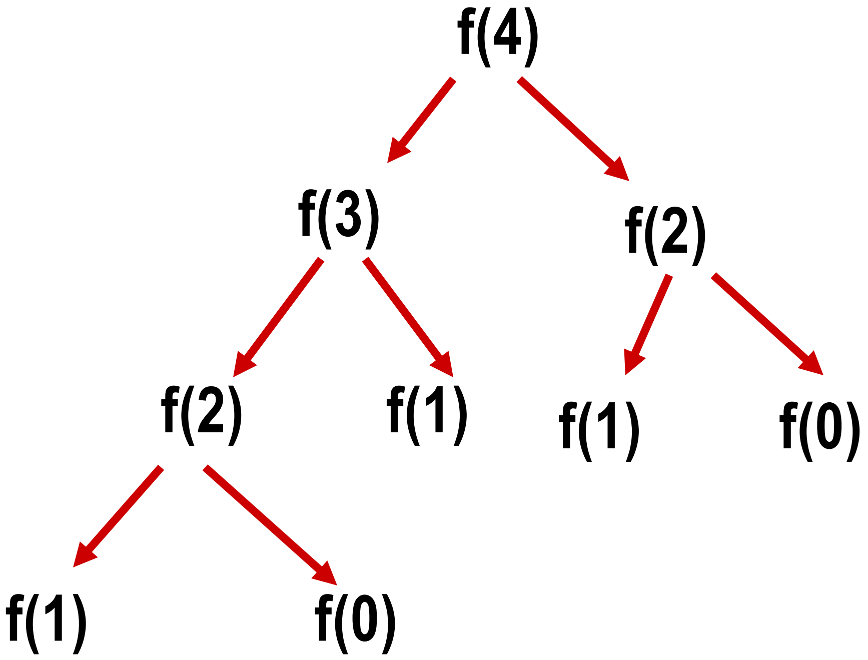
Tracing Execution

```
int   f (int n)
{
     if  (n  < 2)   return (n);
     else  return ( f(n – 1) + f(n – 2) );
}
```

How many times is the function called when evaluating f(4) ?

Inefficiency:

- Same thing is computed several times.



called 9 times

Notable Point

- Every recursive program can also be written without recursion
- Recursion is used for programming convenience, not for performance enhancement
- Sometimes, if the function being computed has a nice recurrence form, then a recursive code may be more readable

Important things to remember

- Think how the whole problem (finding max of n elements in A) can be solved if you can solve the exact same problem on a smaller problem (finding max of first n-1 elements of the array).
- But then, do NOT think how the smaller problem will be solved, just call the function recursively and assume it will be solved.

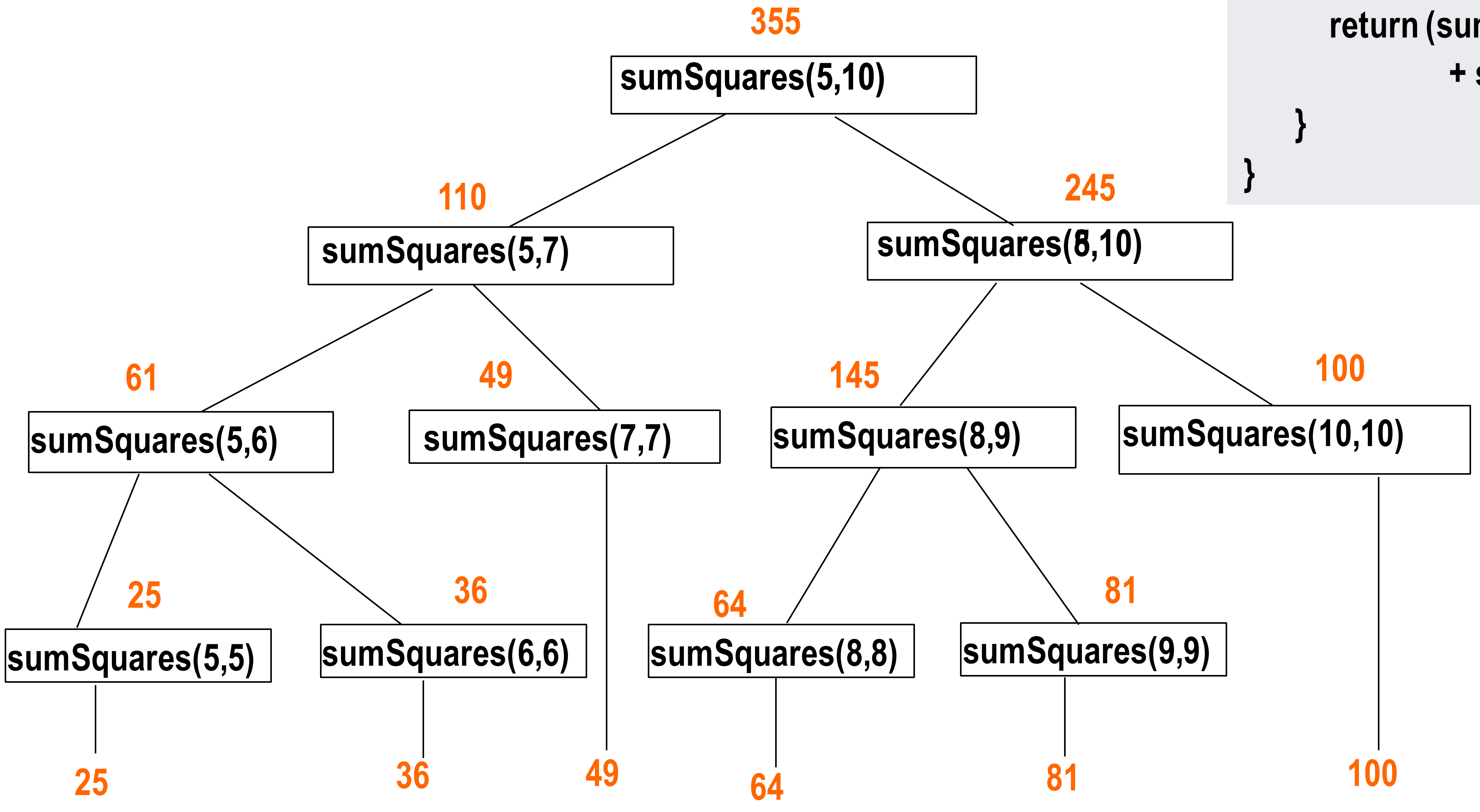
- When you write a recursive function
    - First write the terminating/base condition
    - Then write the rest of the function
    - Always double-check that you have both

Example: Sum of Squares

```
int sumSquares (int m, int n)
{
    int middle ;
    if (m == n) return(m*m);
    else
    {
        middle = (m+n)/2;
        return (sumSquares(m,middle)  + sumSquares(middle+1,n));
     }
}
```

Annotated Call Tree

```
int sumSquares (int m, int n)
{
    int middle ;
    if (m == n) return(m*m);
    else {
        middle = (m+n)/2;
        return (sumSquares(m,middle)
                    + sumSquares(middle+1,n));
     }
}
```



Example: Printing the digits of an Integer in Reverse

Print the last digit, then print the remaining number in reverse

- Ex: If integer is 743, then reversed is print 3 first, then print the reverse of 74

```
void printReversed( int i )
{
    if (i < 10)   {
      printf(“%d\n”, i); return;
    }
    else {
       printf(“%d”, i%10);
       printReversed(i/10);
    }
}
```

Counting Zeros in a Positive Integer

Check last digit from right

- If it is 0, number of zeros = 1 + number of zeroes in remaining part of the number
- If it is non-0, number of zeros = number of zeroes in remaining part of the number

```
int zeros(int number)
{
    if(number<10) return 0;
    if (number%10 == 0)
                 return(1+zeros(number/10));
    else
      return(zeros(number/10));
}
```

Example: Binary Search

- Searching for an element k in a sorted array A with n elements
- Idea:
    - Choose the middle element A[n/2]
    - If k == A[n/2], we are done
    - If k < A[n/2], search for k between A[0] and A[n/2 -1]
    - If k > A[n/2], search for k between A[n/2 + 1] and A[n-1]
    - Repeat until either k is found, or no more elements to search
- Requires less number of comparisons than linear search in the worst case ($\log_2 n$ instead of n)

Binary Search

```
int binsearch(int A[ ], int low, int high, int k)
{
    int mid;
    printf(“low = %d, high = %d\n”, low, high);

    if (low < high) return 0;
    mid = (low + high)/2;
    printf(“mid = %d, A[%d] = %d\n\n”, mid, mid, A[mid]);

    if (A[mid] == k) return 1;
    else {
            if (A[mid] > k)
                return (binsearch(A, low, mid-1, k));
            else
                return(binsearch(A, mid+1, high, k));
            }
}
```

```
int main()
{
    int A[25], n, k, i, found;

    scanf(“%d”, &n);
    for (i=0; i<n; i++) scanf(“%d”, &A[i]);

    scanf(“%d”, &k);

    found = binsearch(A, 0, n-1, k);
    if (found == 1)
            printf(“%d is present in the array\n”, k);
    else
            printf(“%d is not present in the array\n”, k);
}
```

Output

```
8

9 11 14 17 19 20 23 27

21

low = 0, high = 7

mid = 3, A[3] = 17


low = 4, high = 7

mid = 5, A[5] = 20


low = 6, high = 7

mid = 6, A[6] = 23


low = 6, high = 5

21 is not present in the array
```

```
int binsearch(int A[ ], int low, int high, int k)
{
     int mid;
     printf(“low = %d, high = %d\n”, low, high);

     if (low < high) return 0;
     mid = (low + high)/2;
     printf(“mid = %d, A[%d] = %d\n\n”, mid, mid, A[mid]);

     if (A[mid] == k) return 1;
     else {
               if (A[mid] > k)
                    return (binsearch(A, low, mid-1, k));
               else
                    return(binsearch(A, mid+1, high, k));
               }
}
```

```
8
9 11 14 17 19 20 23 27
14
low = 0, high = 7
mid = 3, A[3] = 17

low = 0, high = 2
mid = 1, A[1] = 11

low = 2, high = 2
mid = 2, A[2] = 14

14 is present in the array
```

Static Variables

```
int Fib (int, int);

int main()
{
    int n;
    scanf("%d", &n);
    if (n == 0 || n ==1)
       printf("F(%d) = %d \n", n, 1);
    else
       printf("F(%d) = %d \n", n,
    Fib(n,2));
    return 0;
}
```

```
int Fib(int n, int i)
{
    static int m1, m2;
    int res, temp;
    if (i==2) {m1 =1; m2=1;}
    if (n == i)  res = m1+ m2;
    else
     {   temp = m1;
          m1 = m1+m2;
          m2 = temp;
         res = Fib(n, i+1);
     }
    return res;
}
```

Static variables remain in existence rather than coming and going each time a function is activated

Common Errors in Writing Recursive Functions

Non-terminating Recursive Function (Infinite recursion)

- No base case

```
int badFactorial(int x) {
   return x * badFactorial(x-1);
}
```

- The base case is never reached

```
int badSum2(int x)
{
   if(x==1) return 1;
   return(badSum2(x--));
}
```

```
int anotherBadFactorial(int x) {
   if(x == 0)
      return 1;
   else
      return x*(x-1)*anotherBadFactorial(x-2);
      // When x is odd, base case never reached!!
}
```

Common Errors in Writing Recursive Functions

Mixing up loops and recursion

```
int anotherBadFactorial(int x) {
  int i, fact = 0;
  if (x == 0)
     return 1;
  else {
     for (i=x; i>0; i=i-1) {
            fact = fact + x*anotherBadFactorial(x-1);
     }
     return fact;
  }
}
```

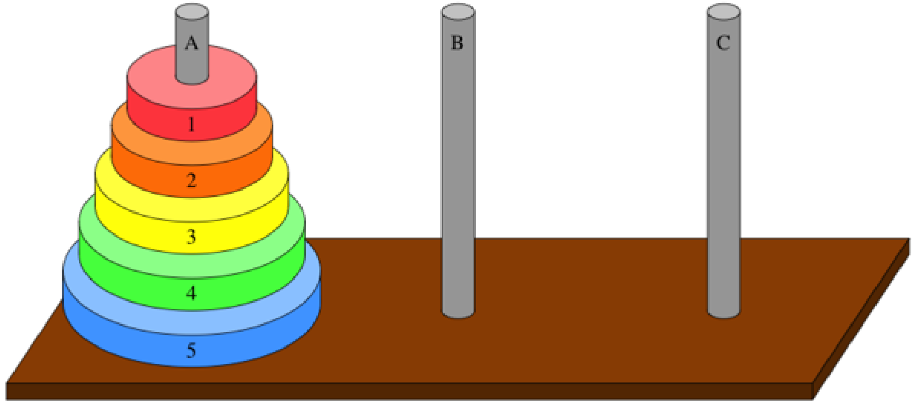
In general, if you have recursive function calls within a loop, think carefully if you need it.

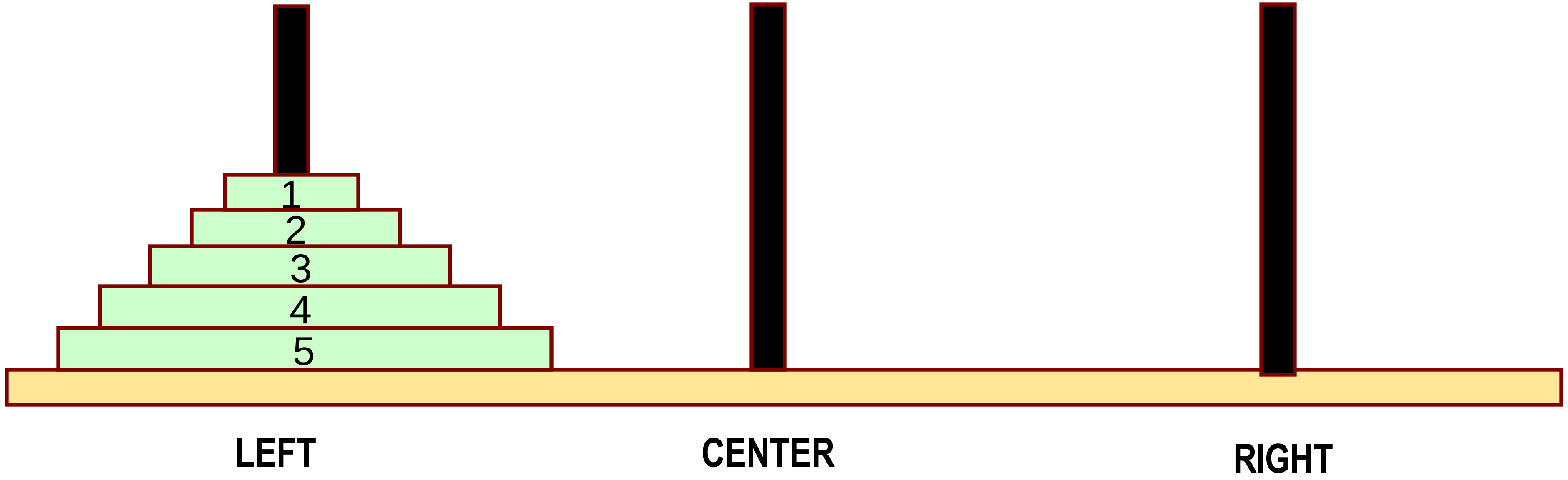
Most recursive functions you will see in this course will not need this

Example :: Towers of Hanoi Problem

The problem statement:

- Initially all the disks are stacked on the LEFT pole.
- Required to transfer all the disks to the RIGHT pole.
    - Only one disk can be moved at a time.
    - A larger disk cannot be placed on a smaller disk.
- CENTER pole is used for temporary storage of disks.

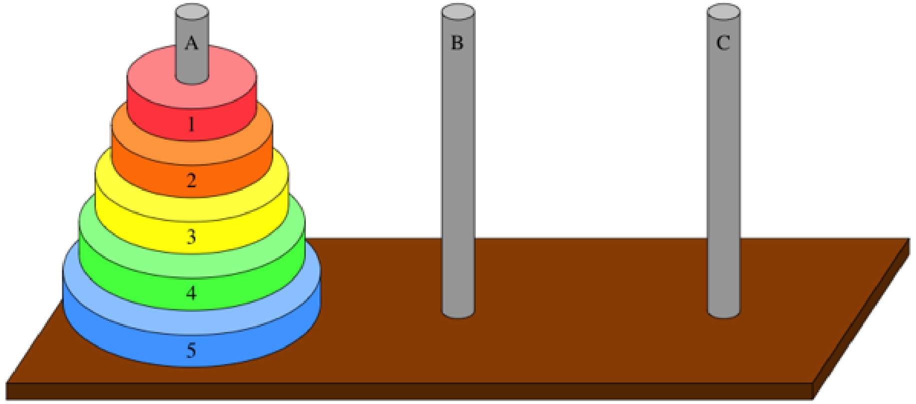




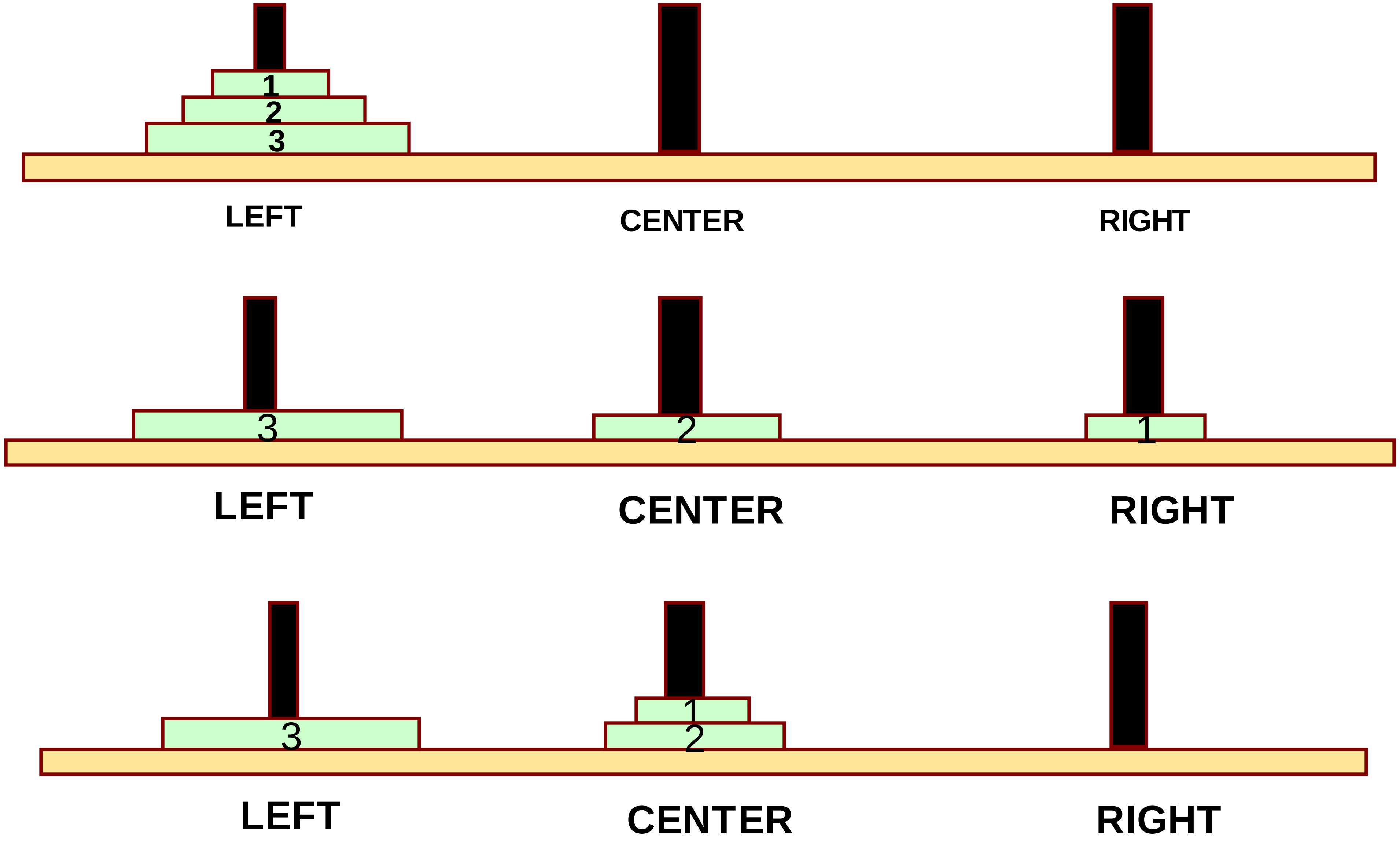
Recursive Formulation

Recursive statement of the general problem of n disks.

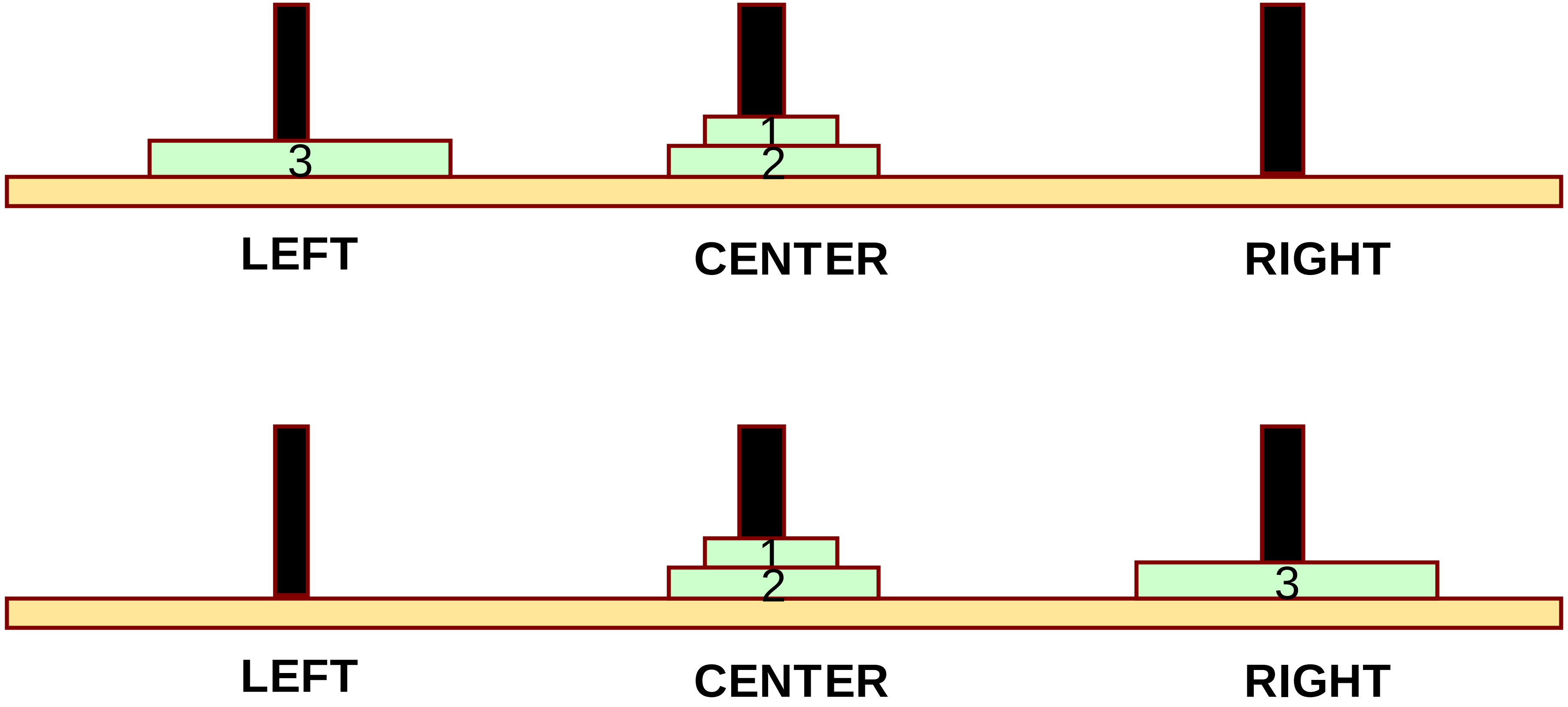
- **Step 1:**
    - Move the top (n-1) disks from LEFT to CENTER.
- **Step 2:**
    - Move the largest disk from LEFT to RIGHT.
- **Step 3:**
    - Move the (n-1) disks from CENTER to RIGHT.



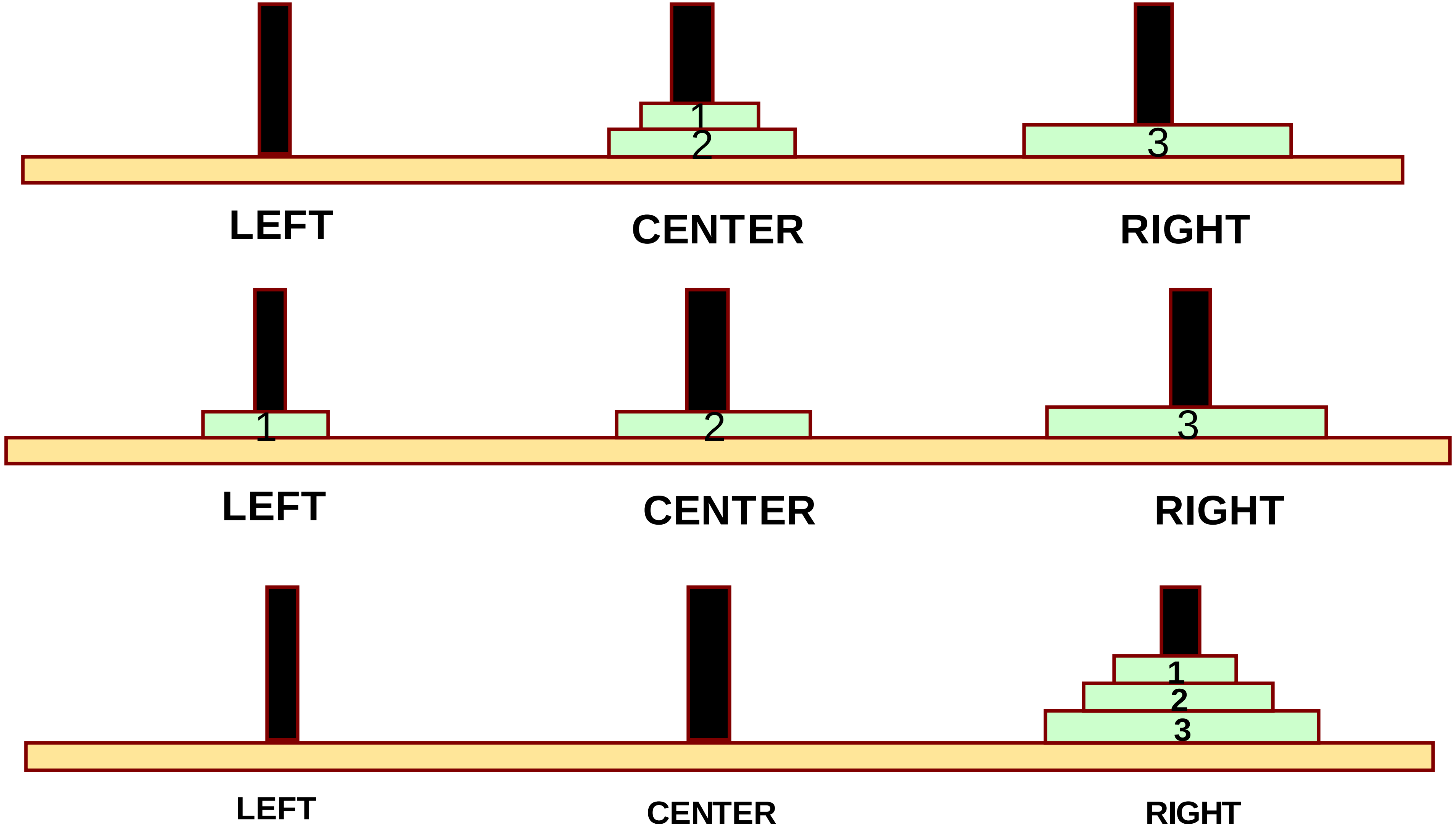
Phase-1: Move top n – 1 from LEFT to CENTER



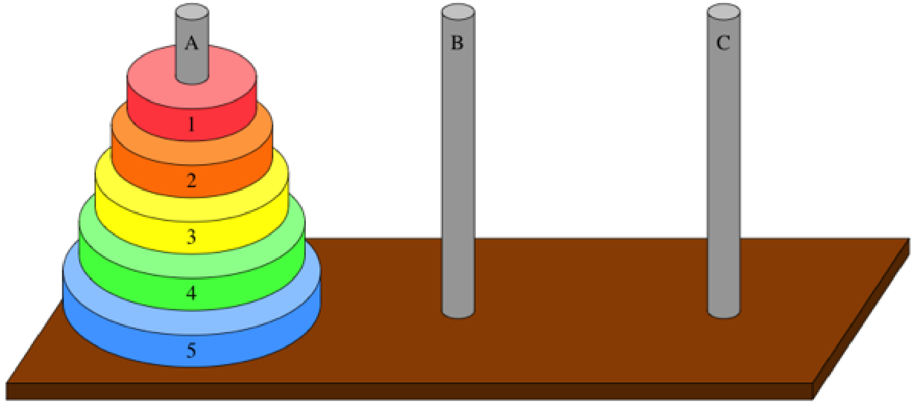
Phase-2: Move the n^{th} disk from LEFT to RIGHT

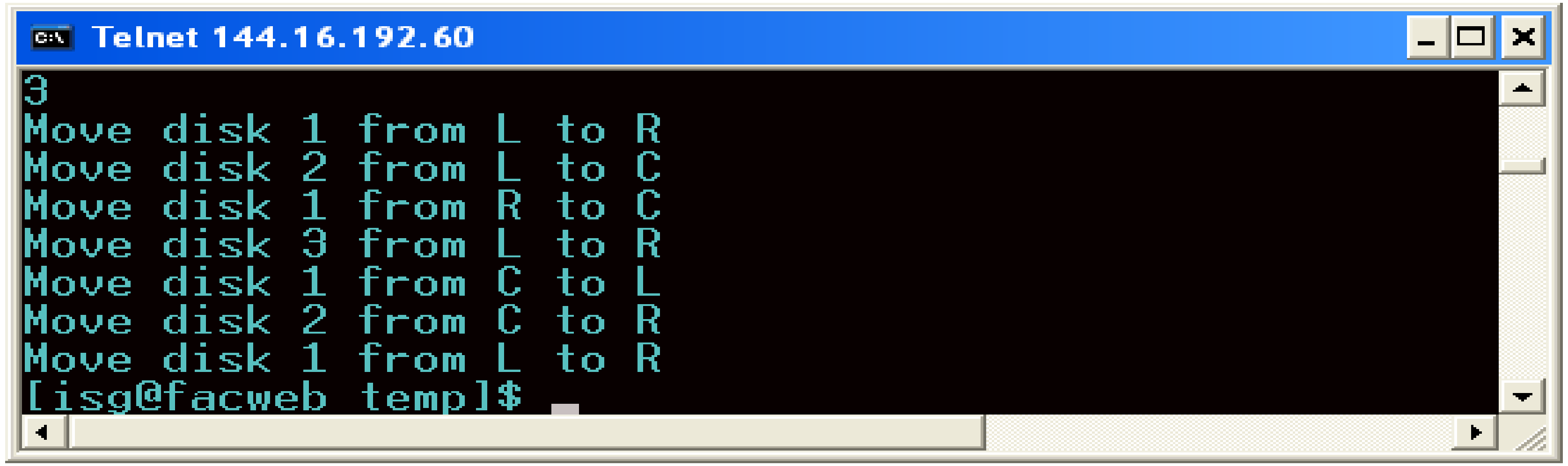


Phase-3: Move top n – 1 from CENTER to RIGHT



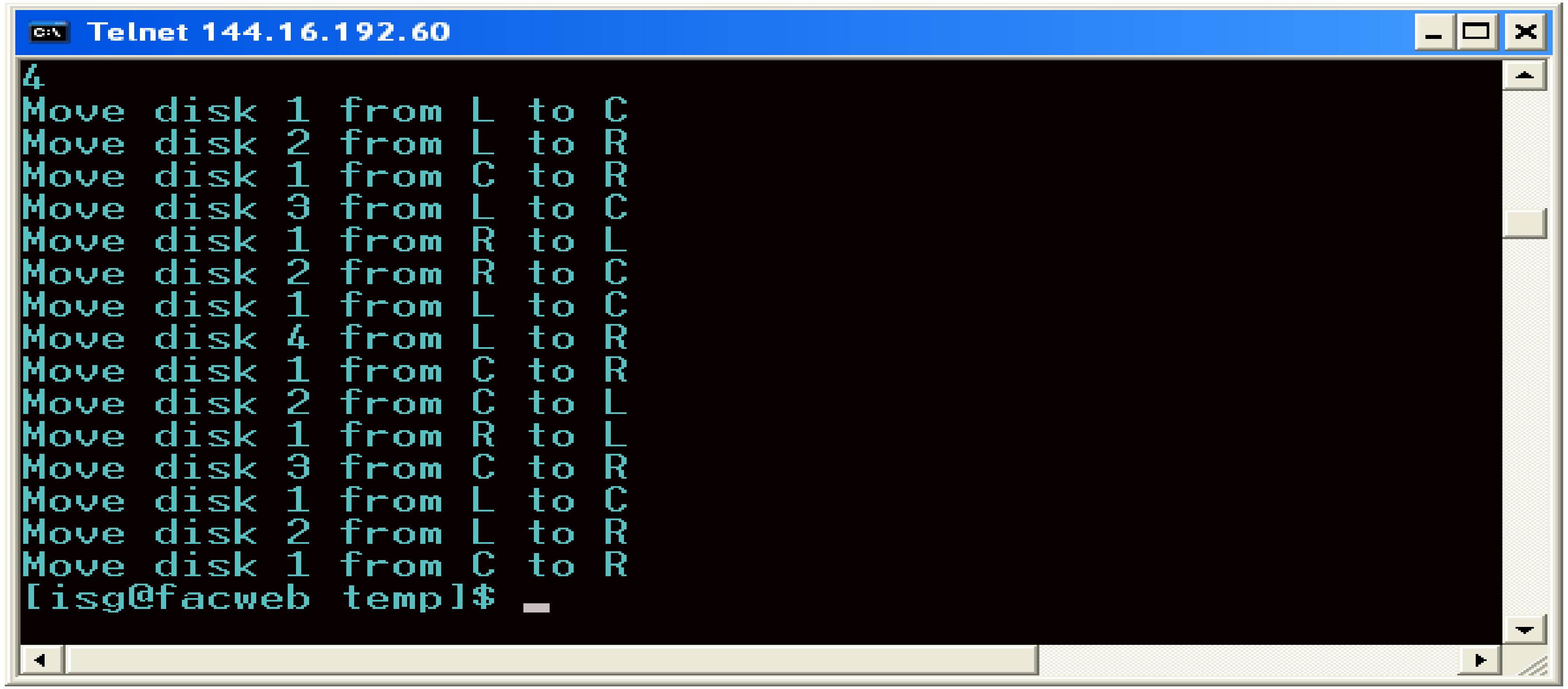
```
#include  <stdio.h>
void  transfer (int n, char from, char to, char temp);

main( )
{               int  n;  /* Number of disks */
                scanf (“%d”, &n);
                transfer (n, ‘L’, ‘R’, ‘C’);
}

void  transfer (int n, char from, char to, char temp)
{
                if  (n > 0)  {
                                transfer  (n-1, from, temp, to);
                                printf (“Move disk %d from %c to %c \n”, n, from, to);
                                transfer (n-1, temp, to, from);
                }
                return;
}
```





```
3
Move disk 1 from L to R
Move disk 2 from L to C
Move disk 1 from R to C
Move disk 3 from L to R
Move disk 1 from C to L
Move disk 2 from C to R
Move disk 1 from L to R
[isg@facweb temp]$ 
```

With 3 discs



```
4
Move disk 1 from L to C
Move disk 2 from L to R
Move disk 1 from C to R
Move disk 3 from L to C
Move disk 1 from R to L
Move disk 2 from R to C
Move disk 1 from L to C
Move disk 4 from L to R
Move disk 1 from C to R
Move disk 2 from C to L
Move disk 1 from R to L
Move disk 3 from C to R
Move disk 1 from L to C
Move disk 2 from L to R
Move disk 1 from C to R
[isg@facweb temp]$ 
```

With 4 discs

Recursion versus Iteration

Repetition

- Iteration: explicit loop
- Recursion: repeated nested function calls

Termination

- Iteration: loop condition fails
- Recursion: base case recognized

Both can have infinite loops

Balance

- Choice between performance (iteration) and good software engineering (recursion).

How are recursive calls implemented?

What we have seen ….

- Activation record gets pushed into the stack when a function call is made.
- Activation record is popped off the stack when the function returns.

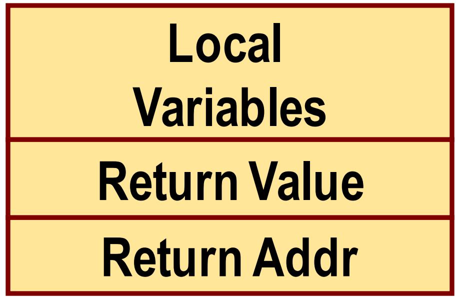
In recursion, a function calls itself.

- Several function calls going on, with none of the function calls returning back.
    - Activation records are pushed onto the stack continuously.
    - Large stack space required.

- Activation records keep popping off, when the termination condition of recursion is reached.

We shall illustrate the process by an example of computing factorial.

- Activation record looks like:



Local Variables

Return Value

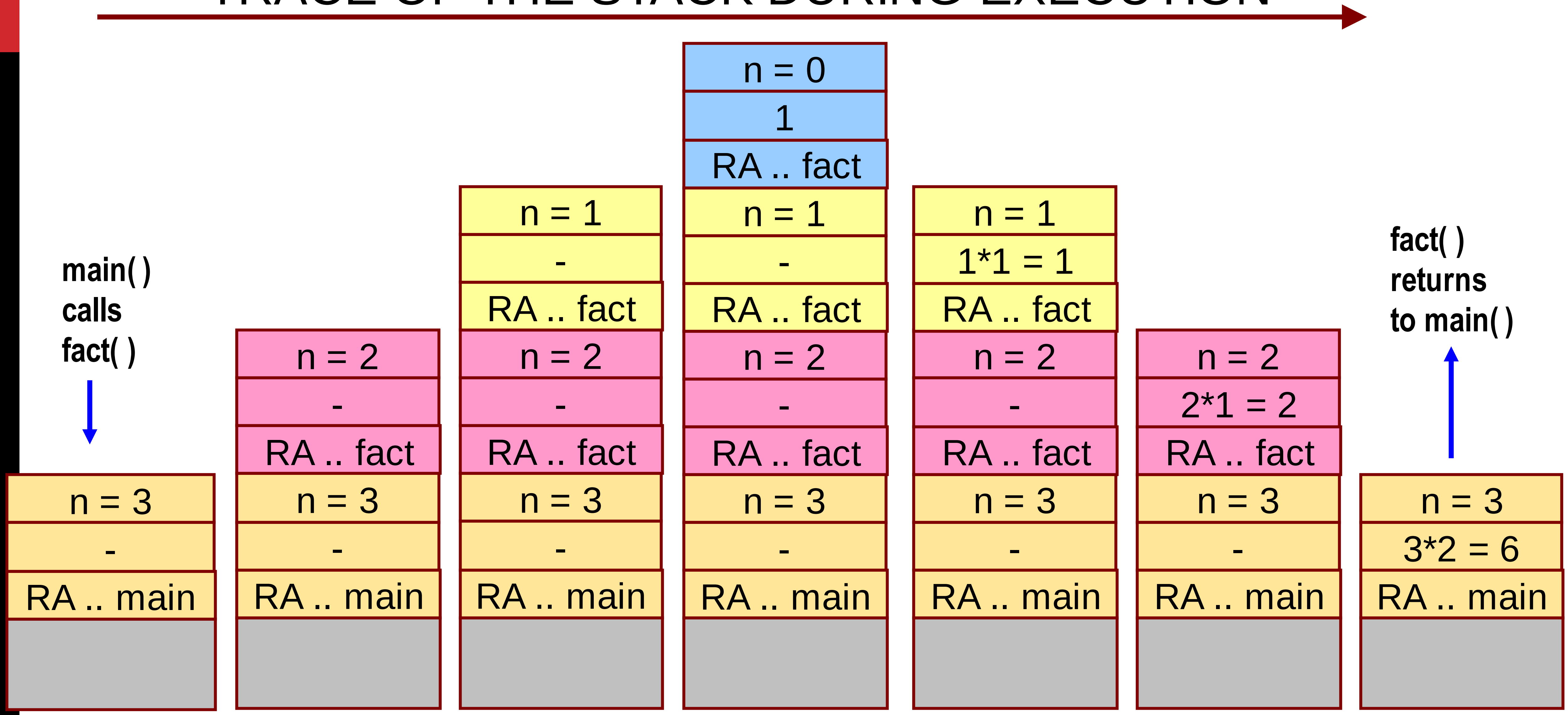
Return Addr

Example:: main( ) calls fact(3)

```
main()
{
   int  n;
   n = 3;
   printf (“%d \n”, fact(n) );
}
```

```
int  fact (n)
int  n;
{
                if   (n = = 0)
                           return (1);
                else
                           return  (n * fact(n-1));
}
```

TRACE OF THE STACK DURING EXECUTION



main( ) calls fact( )

n = 3
-
RA .. main

n = 2
-
RA .. fact
n = 3
-
RA .. main

n = 1
-
RA .. fact
n = 2
-
RA .. fact
n = 3
-
RA .. main

n = 0
1
RA .. fact
n = 1
-
RA .. fact
n = 2
-
RA .. fact
n = 3
-
RA .. main

n = 1
1*1 = 1
RA .. fact
n = 2
-
RA .. fact
n = 3
-
RA .. main

n = 2
2*1 = 2
RA .. fact
n = 3
-
RA .. main

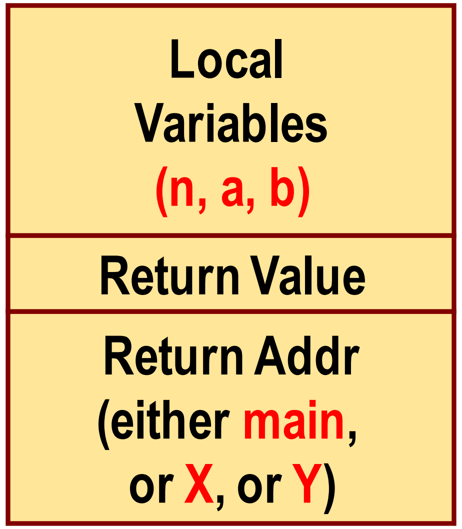
n = 3
3*2 = 6
RA .. main

fact( ) returns to main( )

Do Yourself

Trace the activation records for the following version of Fibonacci sequence.

```
        #include <stdio.h>
        int   f (int n)
        {
            int a, b;
            if  (n  < 2)   return (n);
            else  {
X  →          a = f(n-1);
Y  →          b = f(n-2);
              return (a+b);  }
        }

        main( ) {
           printf("Fib(4) is: %d \n", f(4));
        }
```



Local
Variables
(n, a, b)

Return Value

Return Addr
(either main,
or X, or Y)

Examples

What do the following programs print?

```
void foo(  int n  )
{
      int data;
      if ( n == 0 ) return;
      scanf(“%d”, &data);
      foo ( n – 1 );
      printf(“%d\n”, data);
}
main ( )
{     int k = 5;
      foo ( k );
}
```

```
void foo(  int n  )
{
      int data;
      if ( n == 0 ) return;
      foo ( n – 1 );
      scanf(“%d”, &data);
      printf(“%d\n”, data);
}
main ( )
{     int k = 5;
      foo ( k );
}
```

```
void foo(  int n  )
{
      int data;
      if ( n == 0 ) return;
      scanf(“%d”, &data);
      printf(“%d\n”, data);
      foo ( n – 1 );
}
main ( )
{     int k = 5;
      foo ( k );
}
```

Printing cumulative sum -- *will this work?*

```
int foo(  int n  )
{
      int data, sum ;
      if ( n == 0 ) return 0;
      scanf(“%d”, &data);
      sum = data + foo ( n – 1 );
      printf(“%d\n”, sum);
      return sum;
}
main ( ) {
      int k = 5;
      foo ( k );
}
```

Input: 1 2 3   4 5

Output: 5 9 12 14 15

How to rewrite this so that the output is: 1 3 6 10 15 ?

Printing cumulative sum (two ways)

```
int foo(  int n  )
{
      int data, sum ;
      if ( n == 0 ) return 0;
      sum = foo ( n – 1 );
      scanf(“%d”, &data);
      sum = sum + data;
      printf(“%d\n”, sum);
      return sum;
}
main ( ) {
      int k = 5;
      foo ( k );
}
```

```
Input:     1 2 3   4   5

Output:   1 3 6 10 15
```

```
void foo(  int n, int sum  )
{
      int data ;
      if ( n == 0 ) return 0;
      scanf(“%d”, &data);
      sum = sum + data;
      printf(“%d\n”, sum);
      foo( k – 1, sum ) ;
}
main ( ) {
      int k = 5;
      foo ( k, 0 );
}
```

What does this program print?

```
#include <stdio.h>

int factorial (int n)
{
     static int count=0;
    count++;
    printf (“n=%d, count=%d \n”, n, count);
     if (n == 0) return 1;
    else return (n * factorial(n-1));
}
```

```
main()
{
      int i=6;
      printf (“Value is:  %d \n”, factorial(i));
}
```

What does this program print?

```
#include <stdio.h>
int factorial (int n)
{
    static int count=0;
    count++;
    printf (“n=%d, count=%d \n”, n, count);
     if (n == 0) return 1;
    else return (n * factorial(n-1));
}

main()
{     int i=6;
      printf (“Value is: %d \n”, factorial(i));
}
```

Program output:

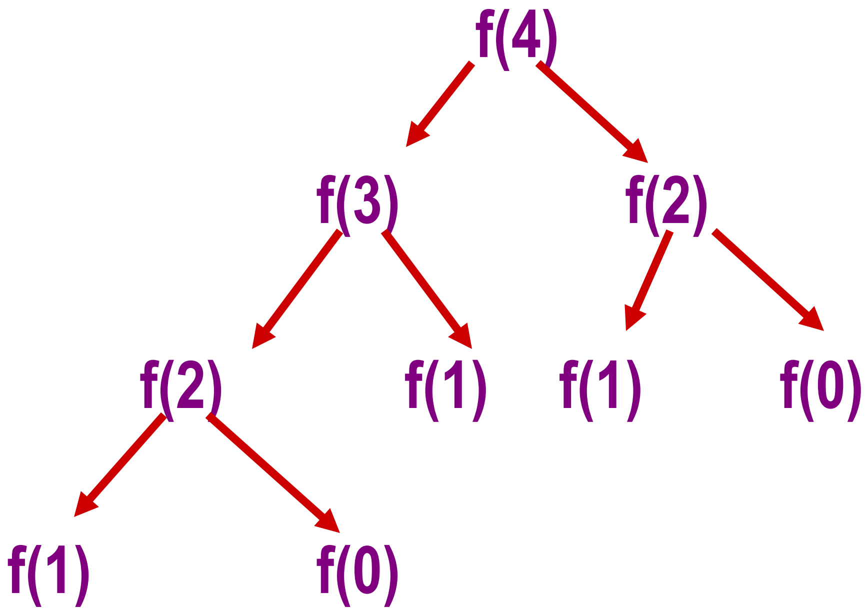
```
n=6, count=1
n=5, count=2
n=4, count=3
n=3, count=4
n=2, count=5
n=1, count=6
n=0, count=7
Value is: 720
```

What does this program print?

```
#include <stdio.h>
int fib (int n)
{
      static int count=0;
      count++;
      printf (“n=%d, count=%d \n”, n, count);
      if (n < 2) return n;
            else return (fib(n-1) + fib(n-2));
}

main( )
{     int i=4;
       printf (“Value is: %d \n”, fib(i));
}
```

What does this program print?

```
#include <stdio.h>
int fib (int n)
{
     static int count=0;
     count++;
     printf (“n=%d, count=%d \n”, n, count);
     if (n < 2) return n;
               else return (fib(n-1) + fib(n-2));
}

main( )
{    int i=4;
     printf (“Value is: %d \n”, fib(i));
}
```



```
n=4, count=1
n=3, count=2
n=2, count=3
n=1, count=4
n=0, count=5
n=1, count=6
n=2, count=7
n=1, count=8
n=0, count=9
Value is: 3 [0,1,1,2,3,5,8,….]
```

Merge-Sort

```
void mergesort ( int a[ ], int lo, int hi )
{
        int m;
        if (lo<hi) {
                m=(lo+hi)/2;
                mergesort(a, lo, m);
                mergesort(a, m+1, hi);
                merge(a, lo, m, hi);
        }
}
```

Function Merge

```
void merge ( int a[ ], int lo, int m, int hi )
{
        int i, j, k;

        // copy both halves of a to auxiliary array b
        for (i=lo; i<=hi; i++) b[i]=a[i];

        i=lo; j=m+1; k=lo;
        // copy back next-greatest element at each time
        while (i<=m && j<=hi)
                if (b[i]<=b[j]) a[k++]=b[i++];
                else a[k++]=b[j++];

        // copy back remaining elements of first half (if any)
        while (i<=m) a[k++]=b[i++];
}
```

Recursive Permutation Generator

```
#define SWAP(x, y, t) { (t) == (x); (x) = (y); (y) = (t) }
void perm (char list[ ], int i, int n)
{
        int j, tmp;
        if (i == n) {
                for (j=0; j<=n; j++) printf(“%c”, list[ j ]);
                printf(“\n”);
        }
        else {
                for (j=i; j <= n; j++) {
                        SWAP(list[ i ], list[ j ], tmp);
                        perm(list, i+1, n);
                        SWAP(list[ i ], list[ j ], tmp);
                }
        }
}
```

Transitive Closure

```
Transclosure ( int adjmat[ ][max], int path[ ][max] )
{
         for (i = 0; i < max; i++)
                  for (j = 0; j < max; j++)
                           path[i][j] = adjmat[i][j];

         for (k = 0; k < max; k++)
           for (i = 0; i < max; i++)
              for (j = 0; j < max; j++)
                  if ((path[i][k] == 1)&&(path[k][j] == 1)) path[i][j] = 1;
}
```

Paying with fewest coins

- A country has coins of denomination 3, 5 and 10 respectively.
- We are to write a function canchange(k) that returns –1 if it is not possible to pay a value of k using these coins.
  - Otherwise it returns the minimum number of coins needed to make the payment.

- For example, canchange(7) will return –1.
- On the other hand, canchange(14) will return 4 because 14 can be paid as 3+3+3+5 and there is no other way to pay with fewer coins.

Paying with fewest coins

```
int canchange( int k )
{
        int a = –1;
        if (k==0) return 0;
        if ( ______________ ) return 1;
        if (k < 3)  ____________________ ;

        a = canchange( _______________ ); if (a > 0) return _______________ ;
        a = canchange(k – 5); if (a > 0) return _____________________ ;
        a = canchange( _______________ ); if (a > 0) return ________________ ;
        return –1;
 }
```

Paying with fewest coins

```
int canchange( int k )
{
        int a = –1;
        if (k==0) return 0;
        if ( (k ==3) || (k == 5) || (k == 10) ) return 1;
        if (k < 3)  return –1  ;

        a = canchange( k – 10 ); if (a > 0) return a+1 ;
        a = canchange( k – 5 ); if (a > 0) return a+1 ;
        a = canchange( k – 3 ); if (a > 0) return a+1 ;
        return –1;
 }
```

Practice Problems

1. Write a recursive function to search for an element in an array
2. Write a recursive function to count the digits of a positive integer (do also for sum of digits)
3. Write a recursive function to reverse a null-terminated string
4. Write a recursive function to convert a decimal number to binary
5. Write a recursive function to check if a string is a palindrome or not
6. Write a recursive function to copy one array to another

Note:

- For each of the above, write the main functions to call the recursive function also
- Practice problems are just for practicing recursion, recursion is not necessarily the most efficient way of doing them